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Skew-coninvolutory matrices

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Abstract

We study the properties of skew-coninvolutory ($E\overline{E} = -I$) matrices, and derive canonical forms and a singular value decomposition. We study the matrix function $\psi_S(A) = S\overline{A}^{-1}S^{-1}$, defined on nonsingular matrices and with S satisfying $S\overline{S} = I$ or $S\overline{S} = -I$. We show that every square nonsingular A may be written as $A = XY$ with $\psi_S(X) = X$ and $\psi_S(Y) = Y^{-1}$. We also give necessary and sufficient conditions on when a nonsingular matrix may be written as a product of a coninvolutory matrix and a skew-coninvolutory matrix or a product of two skew-coninvolutory matrices. Moreover, when A is similar to $\overline{A}^{-1}$, or when A is similar to $-\overline{A}^{-1}$, or when A is similar to $\overline{A}$, or when A is similar to $-\overline{A}$, we determine the possible Jordan canonical forms of A for which the similarity matrix may be taken to be skew-coninvolutory.

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1. Introduction and notation

A square complex matrix E is called *coninvolutory* if $E\overline{E} = I$, that is, E is nonsingular and $E^{-1} = \overline{E}$. It is known [2,3] that every coninvolutory matrix E can be written as $E = \mathrm{e}^{\mathrm{i}R}$ for some real matrix R. Moreover, since $E = \overline{E}^{-1}$, then the singular values of E are either 1, or pairs of σ and $1/\sigma$, where $\sigma > 1$.

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We consider the set of matrices A satisfying $A^{-1} = -\overline{A}$. We call such matrix *skew-coninvolutory*. We derive properties and canonical forms for skew-coninvolutory matrices, analogous to known results for coninvolutory matrices.

We let M_n be the set of n-by-n complex matrices. In [4], a linear operator ϕ_S on M_n was defined by $\phi_S(A) = SA^{\mathrm{T}}S^{-1}$, where S is either symmetric or skew-symmetric. It was shown that every nonsingular matrix $A \in M_n$ may be written as $A = XY$, where $\phi_S(X) = X^{-1}$ and $\phi_S(Y) = Y$. Notice that when $S = I$, this is the classical algebraic polar decomposition of A.

We consider the analogous function on the set of all nonsingular matrices in M_n defined by $\psi_S(A) = S\overline{A}^{-1}S^{-1}$ for some nonsingular S, and show that if we put the restriction $\psi_S(\psi_S(A)) = A$ for all nonsingular $A \in M_n$, then S may be chosen to be either coninvolutory or skew-coninvolutory. We give some properties of ψ_S and prove a ψ_S-polar decomposition for nonsingular matrices. We also determine the respective Jordan canonical forms of ψ_S-orthogonal, ψ_S-symmetric and ψ_S-skew-symmetric matrices.

2. Properties and canonical forms

Definition 1. Let n be a positive integer. We denote the set of skew-coninvolutory matrices by

$$\mathscr{D}_n \equiv \{A \in M_n : A\overline{A} = -I\},$$

and we denote the set of coninvolutory matrices in M_n by $\mathscr{C}_n$. We also set $\mathscr{E}_n \equiv \mathscr{C}_n \cup \mathscr{D}_n$.

Notice that $\mathscr{D}_n$ is empty when n is odd since $\det(A\overline{A})$ is nonnegative for any $A \in M_n$. When n is even, say $n = 2k$, then $\mathscr{D}_n$ is nonempty as $J \equiv \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \in \mathscr{D}_n$.

The following can be verified easily.

Proposition 2. *Let $A \in M_n$ be given. Any two of the following implies the third*:

(a) *A is unitary*.
(b) *A is skew-symmetric*.
(c) *A is skew-coninvolutory*.

Given $A \in \mathscr{D}_n$ and any nonsingular $X \in M_n$, notice that

$$(XA\overline{X}^{-1})\overline{(XA\overline{X}^{-1})} = XA\overline{A}X^{-1} = -I_n.$$

That is, the set $\mathscr{D}_n$ is closed under consimilarity and in particular, real similarity.

Proposition 3. *Let $A \in \mathscr{D}_n$ be given. Then $XA\overline{X}^{-1} \in \mathscr{D}_n$ for any nonsingular $X \in M_n$. In particular, RAR^{-1} and EAE are skew-coninvolutory matrices for real nonsingular R and coninvolutory E.*

It is natural to ask if two similar skew-coninvolutory matrices are also real similar (similar via a real matrix). Recall that two matrices A and $B \in M_n$ are real similar if and only if there exists a nonsingular $S \in M_n$ such that $A = SBS^{-1}$ and $\overline{A} = S\overline{B}S^{-1}$ [3, Theorem 1.1].

Proposition 4. *Two matrices $A, B \in \mathscr{D}_n$ are similar if and only if they are similar via a real matrix.*

Proof. Since real similarity implies similarity, it suffices to prove necessity. Suppose $A, B \in \mathscr{D}_n$ and suppose $A = SBS^{-1}$. Then $\overline{A} = -A^{-1} = -SB^{-1}S^{-1} = S\overline{B}S^{-1}$. $\square$

Let $E \in \mathscr{D}_n$ be given. Then $E = -\overline{E}^{-1}$. Thus, the Jordan blocks and singular values of E come in special pairs.

Proposition 5. *Let $E \in \mathscr{D}_n$ be given.*

(a) *If $J_k(\lambda)$ is a Jordan block of E with multiplicity l, then $J_k\left(-\frac{1}{\overline{\lambda}}\right)$ is a Jordan block of E with multiplicity l.*
(b) *If $\sigma > 0$ is a singular value of E with multiplicity l, then $\frac{1}{\sigma}$ is a singular value of E with multiplicity l.*
(c) *If $\sigma > 0$ and l is a positive integer, then there is a skew-coninvolutory $F \in M_{2l}$ such that σ and $\frac{1}{\sigma}$ are singular values of F, each with multiplicity l.*

Proof. For (c), let $\sigma > 0$ be given. Notice that $F = \begin{bmatrix} 0 & \sigma I_l \\ -\frac{1}{\sigma} I_l & 0 \end{bmatrix} \in M_{2l}$ is a skew-coninvolutory matrix with σ and $\frac{1}{\sigma}$ as singular values, each with multiplicity l. $\square$

Let $A \in M_n$ be nonsingular. Suppose $B = XAX^{-1}$ is coninvolutory. Then $B = \overline{B}^{-1}$, so that $XAX^{-1} = \overline{XA}^{-1}\overline{X}^{-1}$ and

$$A = X^{-1}(\overline{XA}^{-1}\overline{X}^{-1})X = (X^{-1}\overline{X})\overline{A}^{-1}(X^{-1}\overline{X})^{-1}.$$

Notice that $S \equiv X^{-1}\overline{X}$ is coninvolutory. Thus, if A is similar to a coninvolutory matrix, then A is similar to $\overline{A}^{-1}$ via a coninvolutory matrix. One checks that the converse holds as well: if A is similar to $\overline{A}^{-1}$ via a coninvolutory matrix, then A is similar to a coninvolutory matrix. Moreover, the same can be said when we replace coninvolutory with skew-coninvolutory.

Proposition 6. *Let $A \in M_n$ be nonsingular. Then* (i) *A is similar to a coninvolutory matrix if and only if there exists $S \in \mathscr{C}_n$ such that $A = S(\overline{A}^{-1})S^{-1}$; and* (ii) *if n is even, A is similar to a skew-coninvolutory matrix if and only if there exists $S \in \mathscr{C}_n$ such that $A = S(-\overline{A}^{-1})S^{-1}$.*

Let $E \in \mathscr{D}_n$ be given. Then Proposition 5 guarantees that if $J_k(\lambda)$ is a Jordan block of E with multiplicity l, then $J_k\left(-\frac{1}{\overline{\lambda}}\right)$ is a Jordan block of E with multiplicity l. Since, $\lambda \neq -\frac{1}{\overline{\lambda}}$ for any $\lambda \in \mathbb{C}$, we expect E to be similar to a matrix of the form $A \oplus \overline{A}^{-1}$.

Proposition 7. *A matrix $E \in M_{2n}$ is similar to a skew-coninvolutory if and only if E is similar to $A \oplus -\overline{A}^{-1}$ for some nonsingular $A \in M_n$.*

Proof. Let $B = A \oplus -\overline{A}^{-1}$ and $S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then S is coninvolutory and $B = S(-\overline{B}^{-1})S^{-1}$. By Proposition 6, B is similar to some $E \in \mathscr{D}_{2n}$. Thus, if X is similar to $B = A \oplus -\overline{A}^{-1}$, then X is similar to a skew-coninvolutory.

Conversely, suppose $E \in \mathscr{D}_{2n}$ and let J be the Jordan canonical form of E. If $J_k(\lambda)$ is a Jordan block of E, then so is $J_k\left(-\frac{1}{\overline{\lambda}}\right)$. Since λ cannot be equal to $-\frac{1}{\overline{\lambda}}$ for all $0 \neq \lambda \in \mathbb{C}$, then J may be written as

$$J = \oplus_{i=1}^{m}\left(J_{k_i}(\lambda_i) \oplus J_{k_i}\left(-\frac{1}{\overline{\lambda_i}}\right)\right),$$

where $\sum_{i=1}^{m} k_i = n$. Let $A = \oplus_{i=1}^{m} J_{k_i}(\lambda_i)$. Then $-\overline{A}^{-1} = \oplus_{i=1}^{m} - J_{k_i}(\overline{\lambda_i})^{-1}$ which is similar to $\oplus_{i=1}^{m} J_{k_i}\left(-\frac{1}{\overline{\lambda_i}}\right)$. Hence there exists a nonsingular $X \in M_n$ such that $J = A \oplus X(-\overline{A}^{-1})X^{-1} = (I_n \oplus X)(A \oplus -\overline{A}^{-1})(I_n \oplus X)^{-1}$. Therefore E is similar to $A \oplus -\overline{A}^{-1}$. $\square$

The following matrix, defined in [3], was used to obtain examples of, and canonical forms for, coninvolutory matrices. Let k be a positive integer, let $A, B \in M_k$, and define

$$\mathfrak{E}_{2k}(A, B) \equiv \frac{1}{2}\begin{bmatrix} A+B & -i(A-B) \\ i(A-B) & A+B \end{bmatrix}.$$

For $0 \neq \lambda \in \mathbb{C}$, we let $\mathfrak{D}_{2k}(\lambda) \equiv \mathfrak{E}_{2k}(J_k(\lambda), -J_k(\overline{\lambda})^{-1})$. It is known that $\mathfrak{E}_{2k}(A, B)$ is similar to $\operatorname{diag}(A, B)$ via the unitary, symmetric and coninvolutory $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$.

Lemma 8. *Let $A, B \in M_k$ be given. Then $\mathfrak{E}_{2k}(A, B)$ is skew-coninvolutory if and only if $A\overline{B} = -I$.*

Proof. Computations show that

$$\mathfrak{E}_{2k}(A, B)\overline{\mathfrak{E}_{2k}(A, B)} = \frac{1}{2}\begin{bmatrix} A\overline{B}+B\overline{A} & -i(A\overline{B}-B\overline{A}) \\ i(A\overline{B}-B\overline{A}) & A\overline{B}+B\overline{A} \end{bmatrix}, \tag{1}$$

hence, the lemma follows. $\square$

We use the preceding lemma to obtain a canonical form under real similarity for skew-coninvolutory matrices.

Theorem 9. *Let $E \in \mathscr{D}_n$. Then there exist positive integers $m, k_1, \ldots, k_m$, and scalars $\lambda_1, \ldots, \lambda_m$ with $|\lambda_i| \geqslant 1$ for each $i = 1, \ldots, m$ such that $2\sum_{i=1}^{m} k_i = n$ and E is real similar to*

$$\oplus_{i=1}^{m}\mathfrak{D}_{2k_i}(\lambda_i).$$

Proof. Suppose $E \in \mathscr{D}_n$. By Proposition 5, we have that E is similar to $J = \oplus_{i=1}^{m}\big(J_{k_i}(\lambda_i) \oplus -J_{k_i}(\overline{\lambda}_i)^{-1}\big)$. By Lemma 1(g) of [3], J is similar to $\oplus_{i=1}^{m}\mathfrak{D}_{2k_i}(\lambda_i)$, which is skew-coninvolutory by Lemma 8. Hence, Proposition 4 guarantees that E is real similar to $\oplus_{i=1}^{m}\mathfrak{D}_{2k_i}(\lambda_i)$. $\square$

Note that the matrix $\oplus_{i=1}^{m}\mathfrak{D}_{2k_i}(\lambda_i)$ is determined by the Jordan canonical form of E.

Given a skew-coninvolutory matrix $A \in M_{2n}$, notice that A^2 is a coninvolutory matrix. From [3], we know that a coninvolutory matrix has Jordan blocks of the form

(i) $J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)$, where $\lambda \neq 0$, or

(ii) $J_k(\mathrm{e}^{\mathrm{i}\theta})$, where $\theta \in \mathbb{R}$.

Among the coninvolutory matrices, it is natural to ask which ones have a skew-coninvolutory square root. Note that if $\lambda \neq 0$, then $J_k(\lambda) \oplus J_k\left(-\frac{1}{\lambda}\right)$ is similar to $J_k(\lambda) \oplus -\overline{J_k(\lambda)}^{-1}$, which is similar to a skew-coninvolutory matrix using Proposition 7.

Theorem 10. *Let $A \in \mathscr{C}_{2n}$ be given. Then A has a skew-coninvolutory square root if and only if its Jordan blocks may be arranged in the form $J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)$.*

Proof. Suppose $A \in \mathscr{C}_{2n}$. Let A have Jordan canonical form

$$J = \oplus_{i=1}^{m}\left(J_{k_i}(\lambda_i) \oplus J_{k_i}\left(\frac{1}{\overline{\lambda_i}}\right)\right).$$

We choose $K \equiv \oplus_{i=1}^{m}\left(J_{k_i}(\sqrt{\lambda_i}) \oplus J_{k_i}\left(-\overline{\sqrt{\lambda_i}}^{-1}\right)\right)$, which is similar to a skew-coninvolutory matrix by Proposition 7, that is, $K = XMX^{-1}$ for some nonsingular $X \in M_{2n}$ and a skew-coninvolutory $M \in M_{2n}$. Then $K^2 = XM^2X^{-1}$ is similar to J, which implies A is similar to M^2. Since A and M^2 are both coninvolutory, then by Proposition 4, there exists a real matrix $R \in M_{2n}$ such that $A = RM^2R^{-1}$. Since M is skew-coninvolutory and R is real, then by Proposition 3, RMR^{-1} is skew-coninvolutory. Hence, $A = (RMR^{-1})^2$ has a skew-coninvolutory square root.

Conversely, suppose $A = B^2$ for a skew-coninvolutory B. Then B will have Jordan blocks of the form $J_k(\lambda) \oplus J_k(-\overline{\lambda}^{-1})$, where $\lambda \neq 0$. Hence B^2 will have Jordan blocks of the form $J_k(\lambda^2) \oplus J_k(\overline{\lambda^2}^{-1})$ for $\lambda \neq 0$. □

Let r be a positive integer and set $J_{2r} \equiv \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}$. Note that J_{2r} is skew-coninvolutory. Moreover, $J_{2r}^{-1} = J_{2r}^{\mathrm{T}} = -J_{2r}$. We use this to derive a singular value decomposition for skew-coninvolutory matrices.

Lemma 11. *Let $X \in M_{2r}$ be unitary and skew-symmetric, and let $Y \equiv XJ_{2r}^{-1}$. Then Y is unitary and $X = YJ_{2r} = J_{2r}Y^{\mathrm{T}}$.*

Proof. Since X and J_{2r} are unitary, then so is $Y = XJ_{2r}^{\mathrm{T}}$. Moreover, X and J_{2r} are skew-symmetric and $J_{2r}^{-1} = -J_{2r}$, hence, $YJ_{2r} = X = -X^{\mathrm{T}} = -J_{2r}^{\mathrm{T}}Y^{\mathrm{T}} = J_{2r}Y^{\mathrm{T}}$. □

Given a skew-coninvolutory matrix E, we wish to find a singular value decomposition of the form $E = W\Psi W^{\mathrm{T}}$, where W is unitary and Ψ is of some special form. First, consider a singular value decomposition $E = U\Sigma V$ of a skew-coninvolutory matrix $E \in M_n$. By Proposition 5, Σ has the form

$$\Sigma = \left(\sigma_1 I_{n_1} \oplus \frac{1}{\sigma_1} I_{n_1}\right) \oplus \cdots \oplus \left(\sigma_k I_{n_k} \oplus \frac{1}{\sigma_k} I_{n_k}\right) \oplus I_{n_{k+1}}, \tag{2}$$

with $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 1$ and $\sum_{i=1}^{k} 2n_i = n - n_{k+1}$. We partition the unitary matrix

$$\overline{V}U = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1l} \\ X_{21} & X_{22} & \cdots & X_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ X_{l1} & X_{l2} & \cdots & X_{ll} \end{bmatrix} \tag{3}$$

conformal to Σ, with $X_{11}, X_{22} \in M_{n_1}, \ldots, X_{l-2,l-2}, X_{l-1,l-1} \in M_{n_k}, X_{ll} \in M_{n_{k+1}}$ and $l = 2k + 1$. Since $E = -\overline{E}^{-1}$, then $\overline{V}U\Sigma = -\Sigma^{-1}(\overline{V}U)^{\mathrm{T}}$ and following the argument in Theorem 5 of [3], we get

$$\overline{V}U = \begin{bmatrix} 0 & X_1 \\ -X_1^{\mathrm{T}} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & X_k \\ -X_k^{\mathrm{T}} & 0 \end{bmatrix} \oplus X_{k+1}, \tag{4}$$

where X_{k+1} is unitary and skew-symmetric, and of even dimension n_{k+1}.

For $1 \leqslant i \leqslant k$, we let $\Psi_i = \begin{bmatrix} 0 & -\frac{1}{\sigma_i} I_{n_i} \\ \sigma_i I_{n_i} & 0 \end{bmatrix}$ and $Y_i = \begin{bmatrix} -X_i & 0 \\ 0 & -X_i^{\mathrm{T}} \end{bmatrix}$. Then

$$Y_i \Psi_i = \begin{bmatrix} 0 & X_i \\ -X_i^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \sigma_i I_{n_i} & 0 \\ 0 & \frac{1}{\sigma_i} I_{n_i} \end{bmatrix} = \Psi_i Y_i^{\mathrm{T}}.$$

For $i = k + 1$, Lemma 11 guarantees that there exists a unitary Y_{k+1} such that $Y_{k+1}\Psi_{k+1} = \Psi_{k+1}Y_{k+1}^{\mathrm{T}}$, where $\Psi_{k+1} \equiv -J_{2r}$ and $2r = n_{k+1}$.

Now, Y_i is unitary for $i = 1, \ldots, k, k + 1$, hence, we can find a unitary and polynomial square root for Y_i, say Z_i. Since $Y_i\Psi_i = \Psi_i Y_i^{\mathrm{T}}$, then $Z_i\Psi_i = \Psi_i Z_i^{\mathrm{T}}$, for $i = 1, \ldots, k + 1$. Let $Y \equiv Y_1 \oplus \cdots \oplus Y_k \oplus Y_{k+1}$, let $Z \equiv Z_1 \oplus \cdots \oplus Z_k \oplus Z_{k+1}$, and let $\Psi \equiv \Psi_1 \oplus \cdots \oplus \Psi_k \oplus \Psi_{k+1}$. Then $\overline{V}U\Sigma = Y\Psi = Z^2\Psi = Z(Z\Psi) = Z\Psi Z^{\mathrm{T}}$. Hence, $E = U\Sigma V = V^{\mathrm{T}}(\overline{V}U\Sigma)V = (V^{\mathrm{T}}Z)\Psi (V^{\mathrm{T}}Z)^{\mathrm{T}} = W\Psi W^{\mathrm{T}}$, where $W = V^{\mathrm{T}}Z$ is unitary and

$$\Psi = \begin{bmatrix} 0 & -\frac{1}{\sigma_1} I_{n_1} \\ \sigma_1 I_{n_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -\frac{1}{\sigma_k} I_{n_k} \\ \sigma_k I_{n_k} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -I_r \\ I_r & 0 \end{bmatrix}. \tag{5}$$

Conversely, if Ψ is of the form described in (5) which is skew-coninvolutory, and if W is unitary, then by Proposition 3, $W\Psi W^{\mathrm{T}} = W\Psi\overline{W}^{-1}$ is skew-coninvolutory.

Theorem 12. *Let $E \in \mathscr{D}_{2n}$ be given. Then there exists a unitary $W \in M_{2n}$ and*

$$\Psi = \oplus_{i=1}^{k} \left(\begin{bmatrix} 0 & -\frac{1}{\sigma_i} I_{n_i} \\ \sigma_i I_{n_i} & 0 \end{bmatrix} \right) \oplus \begin{bmatrix} 0 & -I_{n_{k+1}} \\ I_{n_{k+1}} & 0 \end{bmatrix}, \tag{6}$$

with $\sigma_1 > \cdots > \sigma_k > 1$ and $\sum_{i=1}^{k+1} n_i = n$ such that $E = W\Psi W^{\mathrm{T}}$. Conversely $W\Psi W^{\mathrm{T}}$ is skew-coninvolutory whenever W is unitary and Ψ is of the form (6).

Since J_{2k} is skew-coninvolutory, then by Proposition 3, $XJ\overline{X}^{-1}$ is skew-coninvolutory for all nonsingular $X \in M_n$. Let n, m be positive integers. Observe that $J_{2n} \oplus J_{2m}$ is similar to $J_{2(n+m)}$ via a permutation matrix. We use this to show that every skew-coninvolutory matrix is consimilar to J_{2n} for some n.

Theorem 13. *Let $E \in M_{2n}$ be given. Then E is skew-coninvolutory if and only if $E = XJ_{2n}\overline{X}^{-1}$, for some nonsingular $X \in M_{2n}$.*

Proof. Note that for $\sigma > 0$ and a positive integer m,

$$\begin{bmatrix} 0 & -\frac{1}{\sigma} I_m \\ \sigma I_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{\sigma}} I_m \\ \sqrt{\sigma} I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{\sigma}} I_m \\ \sqrt{\sigma} I_m & 0 \end{bmatrix}.$$

Suppose E is skew-coninvolutory. Then by Theorem 12, there exists a unitary U and a skew-coninvolutory Ψ of the form (6) such that $E = U\Psi U^{\mathrm{T}}$. Let $W_i \equiv \begin{bmatrix} 0 & \frac{1}{\sqrt{\sigma_i}} I_{n_i} \\ \sqrt{\sigma_i} I_{n_i} & 0 \end{bmatrix}$, for $i =$

$1, \ldots, k$, $W_{k+1} \equiv \begin{bmatrix} 0 & -I_{n_{k+1}} \\ I_{n_{k+1}} & 0 \end{bmatrix}$ and $W \equiv \oplus_{i=1}^{k+1} W_i$. Then $E = U\Psi U^{\mathrm{T}} = (UW)K(\overline{UW})^{-1}$, where $K = \oplus_{i=1}^{k+1} J_{2n_i}$ is similar to J_{2n} via a permutation matrix P. Hence,

$$E = (UWP)J_{2n}(\overline{UWP})^{-1}. \qquad \square$$

3. The function $\psi_S(A) = S\overline{A}^{-1}S^{-1}$

3.1. Properties of ψ_S

Let M_n^* be the set of nonsingular n-by-n matrices, and let $S \in M_n^*$ be given. We define $\psi_S : M_n^* \to M_n^*$ by $\psi_S(A) = S\overline{A}^{-1}S^{-1}$. A similar function ($\phi_S(A) = SA^{\mathrm{T}}S^{-1}$ defined on M_n) was used in [4] to further study the QS decomposition (orthogonal-symmetric) of a square matrix. We begin with the following.

Lemma 14. *Let $S \in M_n^*$ be given. Then*

(a) $\psi_S(I) = I$.
(b) $\psi_S(AB) = \psi_S(B)\psi_S(A)$ *for any* $A, B \in M_n^*$.
(c) $\psi_S(A^{-1}) = \psi_S(A)^{-1}$ *for any* $A \in M_n^*$.

It is known [4] that if $\phi_S(A) = SA^{\mathrm{T}}S^{-1}$ satisfies $\phi_S(\phi_S(A)) = A$ for all $A \in M_n$, then S must be either symmetric or skew-symmetric. When $S \in \mathscr{E}_n$, then $S\overline{S} = \pm I$ so that $\psi_S(\psi_S(A)) = \psi_S(S\overline{A}^{-1}S^{-1}) = S\overline{S}A(S\overline{S})^{-1} = A$. We now show that if the function $\psi_S(A) = S\overline{A}^{-1}S^{-1}$ satisfies $\psi_S(\psi_S(A)) = A$ for all $A \in M_n^*$ then S may be taken to be an element of $\mathscr{E}_n$.

Proposition 15. *Let $X \in M_n^*$ be given. If $\psi_X(\psi_X(A)) = A$ for all $A \in M_n^*$, then there exists $S \in \mathscr{E}_n$ such that $\psi_X = \psi_S$. Conversely, if $S \in \mathscr{E}_n$, then $\psi_S(\psi_S(A)) = A$ for all $A \in M_n^*$.*

Proof. If $\psi_X(\psi_X(A)) = A$ for all $A \in M_n^*$, then $X\overline{X}A = AX\overline{X}$ for all nonsingular A, and thus, $X\overline{X} = \alpha I$ for some nonzero $\alpha \in \mathbb{C}$ since X is nonsingular. Now, $X = \alpha\overline{X}^{-1} = \alpha\overline{(\alpha\overline{X}^{-1})}^{-1} = \alpha\overline{\alpha}^{-1}X$. Therefore, $\alpha\overline{\alpha}^{-1} = 1$, that is, $\alpha \in \mathbb{R}$. Set $S \equiv |\alpha|^{-1/2}X$ and notice that $\psi_X(A) = (|\alpha|^{1/2}S)\overline{A}^{-1}(|\alpha|^{-1/2}S^{-1}) = S\overline{A}^{-1}S^{-1} = \psi_S(A)$. Moreover,

$$S\overline{S} = \frac{1}{|\alpha|}X\overline{X} = \frac{\alpha}{|\alpha|}I = \begin{cases} I, & \text{if } \alpha > 0, \\ -I, & \text{if } \alpha < 0. \end{cases}$$

Hence $S \in \mathscr{E}_n$, as desired. $\square$

The following definitions are analogs of the terminologies defined in [4].

Definition 16. Let $S \in \mathscr{E}_n$ be given. We say that $A \in M_n^*$ is ψ_S-*symmetric* if $\psi_S(A) = A$; A is called ψ_S -*orthogonal* if $\psi_S(A) = A^{-1}$; and A is called ψ_S-*skew-symmetric* if $\psi_S(A) = -A$.

Let $S \in \mathscr{E}_n$ be given. Then $\psi_S(\psi_S(A)) = A$ for all $A \in M_n^*$. Because $\psi_S(AB) = \psi_S(B)\psi_S(A)$ for any $A, B \in M_n^*$, as well, then $\psi_S(A\psi_S(A)) = A\psi_S(A)$. That is, $A\psi_S(A)$ is ψ_S-symmetric for any $A \in M_n^*$.

Lemma 17. *Let $S \in \mathscr{E}_n$ be given.*

(a) *$A\psi_S(A)$ and $\psi_S(A)A$ are ψ_S-symmetric for any $A \in M_n^*$.*
(b) *If A and B are ψ_S-orthogonal, then AB is ψ_S-orthogonal.*
(c) *Suppose $S \in \mathscr{C}_n$ and suppose that $S_1^2 = S$ with $S_1 \in \mathscr{E}_n$. Then $S_1AS_1^{-1}$ is ψ_S-symmetric if and only if $A \in \mathscr{C}_n$, and $S_1AS_1^{-1}$ is ψ_S-skew-symmetric if and only if $A \in \mathscr{D}_n$.*

Proof. Claims (a) and (b) follow directly from Lemma 14.

For (c), suppose $S \in \mathscr{C}_n$ and suppose that $S_1^2 = S$ with $S_1 \in \mathscr{E}_n$. Then $S_1 = SS_1^{-1} = \pm S\overline{S}_1$. Now,

$$\begin{aligned}\psi_S(S_1AS_1^{-1}) &= S(\overline{S_1}\,\overline{A}^{-1}\overline{S_1}^{-1})S^{-1} \\ &= (\pm S_1)\overline{A}^{-1}(\pm S_1^{-1}) \\ &= S_1\overline{A}^{-1}S_1^{-1}.\end{aligned}$$

Notice that $\psi_S(S_1AS_1^{-1})$ is equal to $S_1AS_1^{-1}$ if and only if $\overline{A}^{-1} = A$; and $\psi_S(S_1AS_1^{-1})$ is equal to $-(S_1AS_1^{-1})$ if and only if $\overline{A}^{-1} = -A$. □

3.2. ψ_S-Skew-symmetric and ψ_S-symmetric matrices

Suppose that $A \in M_n$ is nonsingular and that A is similar to $-\overline{A}^{-1}$. We show that for such a matrix A, the matrix of similarity may be chosen to be coninvolutory or skew-coninvolutory, that is, $A = S(-\overline{A}^{-1})S^{-1}$, with $S \in \mathscr{E}_n$. Note that in this case, $\psi_S(A) = S(\overline{A}^{-1})S^{-1} = -A$, so that A is ψ_S-skew-symmetric.

Theorem 18. *Let $A \in M_n$ be nonsingular. The following are equivalent:*

(a) *A is similar to $-\overline{A}^{-1}$.*
(b) *A is similar to a skew-coninvolutory matrix.*
(c) *A is similar to a skew-coninvolutory via a coninvolutory.*
(d) *A is similar to $-\overline{A}^{-1}$ via a coninvolutory.*
(e) *A is a product of a coninvolutory and a skew-coninvolutory.*
(f) *A is similar to $-\overline{A}^{-1}$ via a skew-coninvolutory.*

Proof. Suppose A is similar to $-\overline{A}^{-1}$. Then A has Jordan canonical form $J = \oplus\left(J_{k_i}(\lambda_i) \oplus J_{k_i}\left(\frac{1}{\lambda_i}\right)\right)$. Set $B \equiv \oplus J_{k_i}(\lambda_i)$ and note that A is similar to $B \oplus \overline{B}^{-1}$, so that by Proposition 7, A is similar to a skew-coninvolutory matrix.

Suppose $A = X^{-1}BX$, where $B \in \mathscr{D}_n$. Write $X = RE$, where R is real and E is coninvolutory. Then $A = E^{-1}R^{-1}BRE$, and by Proposition 3, $R^{-1}BR$ is skew-coninvolutory. Hence, A is similar to a skew-coninvolutory via a coninvolutory.

Suppose $A = E^{-1}CE$, where $E \in \mathscr{C}_n$ and $C \in \mathscr{D}_n$. Then $\overline{A}^{-1} = -ECE^{-1}$, so that $C = E^{-1}(-\overline{A}^{-1})E$. Hence, $A = E^{-2}(-\overline{A}^{-1})E^2$, and note that E^2 is a square of a coninvolutory matrix and hence, is also coninvolutory.

Suppose $A = E^{-1}(-\overline{A}^{-1})E$, where E is coninvolutory. Let

$$Z = E^{-1}\left(-\overline{A}^{-1}\right) = AE^{-1},$$

and observe that $Z\overline{Z} = E^{-1}(-\overline{A}^{-1})\overline{(AE^{-1})} = -I$. Hence, Z is skew-coninvolutory. Moreover, $A = ZE$ is a product of a skew-coninvolutory matrix and a coninvolutory matrix.

Suppose $A = XY$, where X is skew-coninvolutory and Y is coninvolutory. Then $\overline{A}^{-1} = \overline{Y}^{-1}\overline{X}^{-1} = -YX$, which implies that $Y = -\overline{A}^{-1}X^{-1}$. Therefore, $A = X(-\overline{A}^{-1})X^{-1}$ for a skew-coninvolutory X.

One checks that (f) implies (a). □

We now look at a matrix A that is similar to $\overline{A}^{-1}$. We show $A = S\overline{A}^{-1}S^{-1}$ for some coninvolutory S so that A is ψ_S-symmetric.

Theorem 19. *Let $A \in M_n$ be nonsingular. The following are equivalent*:

(a) *A is similar to $\overline{A}^{-1}$.*
(b) *A is similar to a coninvolutory matrix.*
(c) *A is similar to a coninvolutory via a coninvolutory matrix.*
(d) *A is similar to $\overline{A}^{-1}$ via a coninvolutory matrix.*
(e) *A is a product of two coninvolutory matrices.*

Proof. Suppose A is similar to $\overline{A}^{-1}$. If A has Jordan canonical form J, then J is a direct sum of blocks of the form $J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)$ and $J_k(\mathrm{e}^{\mathrm{i}\theta})$ for $\theta \in \mathbb{R}$. Thus, J is similar to a coninvolutory matrix and therefore, so is A.

Suppose $A = X^{-1}BX$ for some nonsingular $X \in M_n$ and a coninvolutory $B \in M_n$. Write $X = RE$, where R is real and E is coninvolutory E. Then $A = E^{-1}R^{-1}BRE$ and notice that $R^{-1}BR$ is also coninvolutory. Therefore A is similar to a coninvolutory via a coninvolutory.

If $A = X^{-1}EX$ for coninvolutory matrices X and E, then $\overline{A}^{-1} = XEX^{-1}$, so that $E = X^{-1}\overline{A}^{-1}X$ and $A = (X^2)^{-1}\overline{A}^{-1}X^2$. Note that X^2 is coninvolutory.

If $A = E^{-1}\overline{A}^{-1}E$ for a coninvolutory E, then $E^{-1}\overline{A}^{-1} = AE^{-1}$ is coninvolutory. Therefore A is a product of two coninvolutories.

If $A = XY$ for coninvolutory matrices X and Y, then $\overline{A}^{-1} = (\overline{XY})^{-1} = \overline{Y}^{-1}\overline{X}^{-1} = YX$. Hence $Y = \overline{A}^{-1}X^{-1}$ and $A = X\overline{A}^{-1}X^{-1}$, that is, A is similar to $\overline{A}^{-1}$. □

The following theorem gives a necessary and sufficient condition for A to be similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix.

Theorem 20. *Let $A \in M_{2n}$ be given. Then A is similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix if and only if A is a product of two skew-coninvolutory matrices.*

Proof. Suppose $A = XY$ for skew-coninvolutory matrices X, Y. Then $\overline{A}^{-1} = \overline{Y}^{-1}\overline{X}^{-1} = YX$, hence $Y = \overline{A}^{-1}X^{-1}$. Therefore, $A = X\overline{A}^{-1}X^{-1}$, where X is skew-coninvolutory. Conversely, suppose $A = X\overline{A}^{-1}X^{-1}$ for a skew-coninvolutory X, and observe that $(\overline{A}^{-1}X^{-1})(\overline{\overline{A}^{-1}X^{-1}}) = -I$. Hence, A is a product of two skew-coninvolutories. □

3.3. ψ_S-Polar decomposition

Every nonsingular matrix A may be written as $A = XY$, where X is orthogonal and Y is symmetric. We now show that every nonsingular matrix A may be written as $A = XY$, where X is ψ_S-orthogonal and Y is ψ_S-symmetric. We make use of the following result that shows that every ψ_S-symmetric matrix has a square root that is also ψ_S-symmetric.

Lemma 21. *Let $S \in \mathscr{E}_n$ be given and let $A \in M_n$ be ψ_S-symmetric. Then there exists a ψ_S-symmetric $B \in M_n$ such that $B^2 = A$.*

Proof. Suppose A is ψ_S-symmetric. Then A is similar to $\overline{A}^{-1}$ and, by Theorem 19, there exist matrices $X, E \in \mathscr{C}_n$ such that $A = XEX^{-1}$. Furthermore, Theorem 1.4 in [3] guarantees that E has a polynomial square root F, that is, $F^2 = E$ and $F = p(E)$ for some polynomial $p(t)$. Set $B \equiv XFX^{-1}$ so that $B^2 = A$. Since A is ψ_S-symmetric, we have

$$XEX^{-1} = S\overline{(XEX^{-1})}^{-1}S^{-1} = SX^{-1}EXS^{-1},$$

so that $Xp(E)X^{-1} = SX^{-1}p(E)XS^{-1}$, that is,

$$B = XFX^{-1} = SX^{-1}FXS^{-1} = S\overline{(XFX^{-1})}^{-1}S^{-1} = \psi_S(B),$$

that is, B is a ψ_S-symmetric as desired. □

Let $A \in M_n$ be nonsingular. Then A may be written as $A = RE$, where R is real and E is coninvolutory. Let $S \in \mathscr{E}_n$ be given. Theorem 19 and Lemma 21 imply that $\psi_S(A)A = Y^2$, with Y a ψ_S-symmetric matrix. Define $X \equiv AY^{-1}$. Then

$$\begin{aligned}\psi_S(X) &= \psi_S(AY^{-1})\\ &= \psi_S(Y)^{-1}\psi_S(A)\\ &= Y^{-1}\psi_S(A)\\ &= Y^{-1}(Y^2A^{-1})\\ &= YA^{-1}\\ &= X^{-1}.\end{aligned}$$

Thus, X is ψ_S-orthogonal, Y is ψ_S-symmetric and $A = XY$. We have proven the following.

Theorem 22. *Let $A \in M_n$ be nonsingular and let $S \in \mathscr{E}_n$. Then there exist $X, Y \in M_n$ such that X is ψ_S-orthogonal, Y is ψ_S-symmetric and $A = XY$.*

4. Jordan canonical forms

Let $A \in M_n$ be nonsingular. If A is similar to $-\overline{A}^{-1}$, then Theorem 18 guarantees that the matrix of similarity may be chosen to be coninvolutory or skew-coninvolutory. If A is similar to $\overline{A}^{-1}$, then Theorem 19 ensures that the matrix of similarity may be taken to be coninvolutory. We consider three classes of matrices: A similar to $\overline{A}^{-1}$, A similar to $\overline{A}$ and A similar to $-\overline{A}$. In all three cases, we prove that the matrix of similarity may always be taken to be coninvolutory. However, in each case, we show that the matrix of similarity may be chosen to be skew-coninvolutory only when the Jordan blocks of A occur in pairs with a particular form.

One key observation is that an upper triangular matrix cannot be skew-coninvolutory. This is because if X is upper triangular, then the diagonal entries of the product $X\overline{X}$ are of the form $|x_{ii}|^2$ for some $x_{ii} \in \mathbb{C}$, and thus cannot be equal to -1.

The matrix $J_2(1)$ is similar to $\mathscr{J}_1 \equiv \overline{J_2(1)}^{-1}$ and $\mathscr{J}_2 \equiv \overline{J_2(1)}$ via the coninvolutory matrices $\text{diag}(1, -1)$ and I_2, respectively. If $X \in M_2$ satisfies $J_2(1) = X\mathscr{J}_k X^{-1}$, for $k = 1, 2$, then computations show that X must be upper triangular, and thus cannot be skew-coninvolutory. Therefore, $J_2(1)$ cannot be similar to its conjugate-inverse nor to its conjugate via a skew-coninvolutory matrix.

Similarly, the matrix $J_2(i)$ is similar to $-\overline{J_2(i)}$ via the coninvolutory matrix $\text{diag}(1, -1)$. Again, the matrix of similarity here cannot be chosen to be skew-coninvolutory.

4.1. *A similar to* $\overline{A}^{-1}$

The following lemma is easily verified and gives the result of conjugating a matrix $A \in M_{2k}$ by the skew-coninvolutory matrix $J_{2k} = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$.

Lemma 23. *Let $A = [A_{ij}] \in M_{2k}$, where $A_{ij} \in M_k$ and $i, j = 1, 2$. Then*

$$J_{2k} A J_{2k}^{-1} = \begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix} = J_{2k}^{-1} A J_{2k}.$$

We wish to characterize the matrices A which are similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix by determining their Jordan canonical forms. We first give a class of matrices satisfying Theorem 20.

Theorem 24. *Let $A \in M_n$ have Jordan canonical form*

$$J = \oplus \left(J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)\right).$$

Then A is similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix.

Proof. If all the blocks in the Jordan canonical form of A occur in pairs $J_k(\lambda) \oplus J_k\left(\frac{1}{\overline{\lambda}}\right)$, then A is similar to $J = \oplus(J_k(\lambda) \oplus J_k(\overline{\lambda})^{-1})$, that is, $A = PJP^{-1}$ for some nonsingular P. By Lemma 23,

$$\begin{aligned} J_k(\lambda) \oplus J_k(\overline{\lambda})^{-1} &= J_{2k}(J_k(\overline{\lambda})^{-1} \oplus J_k(\lambda))J_{2k}^{-1} \\ &= J_{2k}\overline{(J_k(\lambda) \oplus J_k(\overline{\lambda})^{-1})}^{-1} J_{2k}^{-1}. \end{aligned}$$

Hence $J = X\overline{J}^{-1}X^{-1}$, where $X = \oplus J_{2k}$ is skew-coninvolutory. Since $A = PJP^{-1}$, then $A = (PX\overline{P}^{-1})\overline{A}^{-1}(PX\overline{P}^{-1})^{-1}$. By Proposition 3, $PX\overline{P}^{-1}$ is skew-coninvolutory, and thus A is similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix. $\square$

A matrix that satisfies Theorem 20 satisfies Theorem 19. But a matrix of even order that is similar to its conjugate-inverse need not satisfy Theorem 20. Theorem 24 shows that a sufficient condition for a matrix A to be similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix is to have its Jordan blocks occur in pairs $J_k(\lambda) \oplus J_k(\overline{\lambda}^{-1})$. To show the necessity of this condition, we consider a Jordan matrix J similar to $\overline{J}^{-1}$. Suppose $E = S^{-1}JS$ such that $E = X^{-1}\overline{E}^{-1}X$ for some skew-coninvolutory X. Then $S^{-1}JS = X^{-1}\overline{S}^{-1}\overline{J}^{-1}\overline{S}X$, which implies $J = (\overline{S}XS^{-1})^{-1}\overline{J}^{-1}(\overline{S}XS^{-1})$, where $\overline{S}XS^{-1}$ is skew-coninvolutory. Conversely, if $E = SJS^{-1}$ is the Jordan canonical form of E and if $J = Y\overline{J}^{-1}Y^{-1}$ for skew-coninvolutory Y, then $Z \equiv SY\overline{S}^{-1}$ is skew-coninvolutory and $E = Z\overline{E}^{-1}Z^{-1}$. Thus it suffices to consider a Jordan matrix J similar to $\overline{J}^{-1}$ via a skew-coninvolutory matrix. We first consider the following technical lemma.

Lemma 25. *Suppose $X = [x_{ij}] \in M_{n,k}$ satisfies*

$$XJ_k(\alpha) = J'X \tag{7}$$

where $J' = \begin{bmatrix} \beta & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & \beta & b_1 & \ddots & \vdots \\ \vdots & & & \ddots & b_2 \\ \vdots & & & \ddots & b_1 \\ 0 & \cdots & & 0 & \beta \end{bmatrix} \in M_n$ *and* $b_1 \neq 0$.

(a) *If $\alpha \neq \beta$, then $X = 0$.*
(b) *Suppose $\alpha = \beta$.*
  (i) *If $k \geqslant n$, then $x_{ij} = 0$ whenever $i + (k - n) > j$.*
  (ii) *If $k < n$, then $x_{ij} = 0$ whenever $i > j$.*
  That is,
  if $k \geqslant n$, then $X = [0\ P]$, where $P \in M_n$ is upper triangular; and
  if $k < n$, then $X = \begin{bmatrix} P \\ 0 \end{bmatrix}$, where $P \in M_k$ is upper triangular.

Proof. A computation reveals that (7) holds if and only if

$$x_{i,j-1} + \alpha x_{ij} = \beta x_{ij} + \sum_{m=1}^{n-1} b_m x_{i+m,j} \tag{8}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$, where we adopt the convention that $x_{pq} = 0$ if $p = 0$, $q = 0$ or $p > n$.

(a) Suppose $\alpha \neq \beta$. Write $X = [x_1 \quad x_2 \quad \cdots \quad x_k]$. Then $XJ_k(\alpha) = [\alpha x_1 \quad x_1 + \alpha x_2 \quad \cdots \quad x_{k-1} + \alpha x_k]$ and $J'X = [J'x_1 \quad J'x_2 \quad \cdots \quad J'x_k]$. Hence, $J'x_1 = \alpha x_1$ and $(J' - \alpha I)x_1 = 0$. Since α is not an eigenvalue of J', then $J' - \alpha I$ is nonsingular, hence, x_1 must be zero. Now, $J'x_2 = x_1 + \alpha x_2 = \alpha x_2$, so that x_2 is also zero. Repeating this process yields $x_i = 0$ for $i = 1, \ldots, k$, and thus, $X = 0$.

(b) Suppose $\alpha = \beta$. Then (8) is equivalent to

$$x_{i,j-1} = \sum_{m=1}^{n-1} b_m x_{i+m,j} \tag{9}$$

for all $i = 1, \ldots, n$, and $j = 1, \ldots, k$. Examining (9) for $i = n$ shows that $x_{n,j-1} = 0$ for all $j = 1, \ldots, k$, that is, $x_{nj} = 0$ for $j = 1, \ldots, k - 1$. This will imply that $x_{n-1,j-1} = b_1 x_{nj} = 0$ for

$j = 1, \ldots, k-1$ and hence $x_{n-1,j} = 0$ for $j = 1, \ldots, k-2$. Repeating this for $i = n-2$ up to $i = k$ implies that all the entries below $x_{n-r,k-r}$, where $r = 1, \ldots, k-1$, are zero. Hence, if $k = n$, then X is upper triangular; if $k > n$, then $X = [0\ X_1]$, where $X_1 \in M_n$ is upper triangular; and if $k < n$, then $X = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$, where $X_3 \in M_k$ is upper triangular. Therefore, $x_{ij} = 0$ if $i + k - n > j$ for all positive integers k and n. This proves (i).

To prove (ii), notice that since $x_{ij} = 0$ whenever $i > j + (n-k)$, then (9) is equivalent to

$$x_{i,j-1} = \sum_{m=1}^{n-k+j} b_m x_{i+m,j} \tag{10}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$. When $j = 1$, (10) becomes

$$0 = \sum_{m=1}^{n-k+1} b_m x_{i+m,1}. \tag{11}$$

Since $x_{i1} = 0$ whenever $i > (n-k)+1$, then (11) becomes $b_1 x_{n-k+1,1} = 0$ when $i = n-k$. Hence $x_{n-k+1,1} = 0$. Hence, if all the entries below $x_{i+1,1}$ are zero for a particular i, then (11) becomes $b_1 x_{i+1,1} = 0$ thus $x_{i+1,1} = 0$. Therefore $x_{i1} = 0$ for $i = 2, \ldots, n$. Suppose that for all $q = 1, \ldots, j-1$, $x_{iq} = 0$ whenever $i > q$. Then (10) implies that

$$\sum_{m=1}^{n-k+j} b_m x_{i+m,j} = 0. \tag{12}$$

If $i = n-k+j-1$, then the sum (12) is just $b_1 x_{n-k+j,j} = 0$, hence $x_{n-k+j,j} = 0$. Taking the sum (12) starting from row $n-k+j-1$ up to row $i = n-k+j-(n-k) = j$ will yield $b_1 x_{i+1,j} = 0$, hence $x_{i+1,j} = 0$. Therefore $x_{i+1,j} = 0$ for $i = j, j+1, \ldots, n-k+j-1$, that is, $x_{ij} = 0$ whenever $i > j$. $\square$

For $\mu \neq 0$,

$$\overline{J_n(\mu)}^{-1} = \begin{bmatrix} \overline{\mu}^{-1} & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & \overline{\mu}^{-1} & b_1 & \ddots & \vdots \\ \vdots & & & \ddots & b_2 \\ \vdots & & & \ddots & b_1 \\ 0 & \cdots & & 0 & \overline{\mu}^{-1} \end{bmatrix},$$

where $b_m = (-1)^m (\overline{\mu})^{-(m+1)}$. Hence, if $\mu \neq 0$ and $A = [a_{ij}] \in M_{n,k}$ such that $AJ_k(\lambda) = \overline{J_n(\mu)}^{-1} A$, then we get a special case of Eq. (7). Thus we have the following assertion.

Lemma 26. *Let $J \in M_n$ such that $J = \oplus_{i=1}^m J_{\lambda_i}$, where $J_{\lambda_i} \in M_{n_i}$,*

$$J_{\lambda_i} = \begin{cases} \oplus_j J_{k_{ij}}(\lambda_i), & \text{if } |\lambda_i| = 1, \\ \oplus_j \left[J_{k_{ij}}(\lambda_i) \oplus J_{k_{ij}}\left(\frac{1}{\overline{\lambda_i}}\right) \right], & \text{if } |\lambda_i| \neq 1, \end{cases}$$

the λ_i's are distinct and $\lambda_i \overline{\lambda_j} \neq 1$ for $i \neq j$. If $X \in M_n$ is such that $XJ = \overline{J}^{-1} X$, then $X = \oplus_{i=1}^m X_i$, where $X_i \in M_{n_i}$.

Proof. Partition $X = (X_{ij})$ conformal to J. The equality $XJ = \overline{J}^{-1}X$ implies that $X_{ij}J_{\lambda_j} = \overline{J_{\lambda_i}}^{-1}X_{ij}$. Since the λ's are distinct and since $\lambda_i\overline{\lambda_j} \neq 1$ whenever $i \neq j$, then by Lemma 25, $X_{ij} = 0$ whenever $i \neq j$. Hence $X = \oplus_{i=1}^{m} X_i$, where $X_i \in M_{n_i}$. $\square$

Suppose $E \in M_n$ is similar to $\overline{E}^{-1}$ and let $e^{i\theta_1}, \ldots, e^{i\theta_s}, \lambda_{s+1}, \frac{1}{\lambda_{s+1}}, \ldots, \lambda_t, \frac{1}{\lambda_t}$, with $|\lambda_i| \neq 1$ be the distinct eigenvalues of E. Note that for $|\lambda_i| \neq 1$, if $J_k(\lambda)$ occurs in J then so does $J_k\left(\frac{1}{\lambda}\right)$. Hence, the Jordan canonical form of E may be written as

$$J = \oplus_{j=1}^{s} J_{e^{i\theta_j}} \oplus \oplus_{j=s+1}^{t} J_{\lambda_j},$$

where $J_{e^{i\theta_j}} = \oplus_j J_{k_j}(e^{i\theta_j})$ and $J_{\lambda_j} = \oplus_j \left[J_{k_j}(\lambda_j) \oplus J_{k_j}\left(\frac{1}{\lambda_j}\right)\right]$. If $X \in M_n$ is such that $XJ = \overline{J}^{-1}X$ and X is skew-coninvolutory, then by Lemma 26, $X = \oplus_{j=1}^{t} X_j$ and X_j must be skew-coninvolutory for all j. We consider what happens when there is an unpaired Jordan block $J_k(\lambda)$, where λ is on the unit circle.

Lemma 27. *Let $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and let $J = \oplus_{i=1}^{r} J_{k_i}(\lambda) \in M_n$ be such that there is an unpaired block $J_{k_i}(\lambda)$ for some i. Then J is not similar to $\overline{J}^{-1}$ via a skew-coninvolutory matrix.*

Proof. Suppose that for some skew-coninvolutory $X = [x_{ij}] \in M_n$ we have $XJ = \overline{J}^{-1}X$ and assume that there is an unpaired block of order k. We consider the following possibilities.

(i) All the blocks are of the same size, that is, $J = \oplus_m J_k(\lambda) \in M_{mk}$, where m is necessarily odd since we are assuming that there is an unpaired block $J_k(\lambda)$. Partition $X = (X_{ij})$ conformal to J. Lemma 25 implies that X_{ij} is a k-by-k upper triangular matrix for all $i, j = 1, \ldots, m$. Let $P \in M_{mk}$ be the permutation matrix obtained by interchanging the $(mk - i)$th column with the $(m - i)k$th column of I_{mk}, $i = 1, 2, \ldots, m - 1$. Then

$$P^{\mathrm{T}}XP = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad \text{where } B_{22} \in M_m. \tag{13}$$

Since X is skew-coninvolutory, then so is $P^{\mathrm{T}}XP$ by Proposition 3. Hence B_{22} is skew-coninvolutory. But B_{22} is of odd dimension and thus cannot be skew-coninvolutory.

(ii) There are Jordan blocks with size different from k. There are three cases: all the other Jordan blocks are of order less than k; all the other Jordan blocks are of order greater than k; and there are Jordan blocks of order less than k and greater than k. We prove only the third case since the proofs of the other two cases use similar arguments.

Suppose $J = J_1 \oplus J_2 \oplus J_3$, where

$J_1 = \oplus_{n_i<k} J_{n_i}(\lambda) \in M_r$,
$J_2 = \oplus_{l_j>k} J_{l_j}(\lambda) \in M_s$ and
$J_3 = \oplus_m J_k(\lambda) \in M_{mk}$, where m is necessarily odd.

Partition $X = (X_{ij})$ conformal to J. By Lemma 25,

X_{33} consists of k-by-k upper triangular blocks;
X_{13} consists of n_i-by-k blocks, all of the form $[0\ V]$, where $V \in M_{n_i}$ is upper triangular;
X_{23} consists of l_j-by-k blocks all of the form $\begin{bmatrix} W \\ 0 \end{bmatrix}$, where $W \in M_k$ is upper triangular;
X_{31} consists of k-by-n_i blocks, all of the form $\begin{bmatrix} Y \\ 0 \end{bmatrix}$, where $Y \in M_{n_i}$ is upper triangular; and
X_{32} consists of k-by-l_j blocks, all of the form $[0\ Z]$, where $Z \in M_k$ is upper triangular.

Let $Q = I_r \oplus I_s \oplus P$, where P is the permutation matrix described above. Then

$$Q^{\mathrm{T}} X Q = \begin{bmatrix} X_{11} & X_{12} & X_{13}P \\ X_{21} & X_{22} & X_{23}P \\ P^{\mathrm{T}}X_{31} & P^{\mathrm{T}}X_{32} & P^{\mathrm{T}}X_{33}P \end{bmatrix}, \tag{14}$$

such that

the last m rows of $P^{\mathrm{T}}X_{31}$ are zero;
the last m rows of $P^{\mathrm{T}}X_{32}$ are zero except for every (l_j)th column;
every (l_j)th row of $X_{23}P$ is zero; and
$P^{\mathrm{T}}X_{33}P$ is of the form described in Eq. (13).

Hence, $Q^{\mathrm{T}}XQ$ may be written as $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, where $C_{22} = B_{22} \in M_m$,

$$C_{21} = \begin{bmatrix} 0_{m,r} & 0 & u_1 & \cdots & 0 & u_t & 0_{m,m(k-1)} \end{bmatrix} \quad \text{and} \quad C_{12} = \begin{bmatrix} D \\ U_1 \\ 0_{1,m} \\ \vdots \\ U_t \\ 0_{1,m} \\ B_{12} \end{bmatrix},$$

such that $u_i \in \mathbb{C}^m$, $D \in M_{r,m}$ and $U_i \in M_{l_i-1,m}$.

If X is skew-coninvolutory, then by Proposition 3, so is $Q^{\mathrm{T}}XQ$, thus $C_{21}\overline{C_{12}} + C_{22}\overline{C_{22}} = -I_m$. Since $C_{21}\overline{C_{12}} = 0$, then C_{22} must be skew-coninvolutory. But this is a contradiction since C_{22} is of odd dimension, hence, X cannot be skew-coninvolutory. $\square$

Now, we exclude all matrices with unpaired $J_k(\mathrm{e}^{\mathrm{i}\theta})$.

Lemma 28. *Let J be as in Lemma 26 and suppose that there exists an unpaired block corresponding to $\lambda_1 = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in \mathbb{R}$. Then J is not similar to $\overline{J}^{-1}$ via a skew-coninvolutory matrix.*

Proof. Let J be as in Lemma 26 and such that there is an unpaired block corresponding to λ_1. If $AJ = \overline{J}^{-1}A$, then $A = A_1 \oplus \cdots \oplus A_r$ and notice that A is skew-coninvolutory if and only if each A_i is skew-coninvolutory. From Lemma 27, A_1 is not skew-coninvolutory. Thus J is not similar to $\overline{J}^{-1}$ via a skew-coninvolutory matrix. $\square$

By Theorem 24 and Lemmas 25–28, we have the following theorem.

Theorem 29. *$A \in M_n$ is similar to $\overline{A}^{-1}$ via a skew-coninvolutory matrix if and only the Jordan canonical form of A is $\oplus\left(J_k(\lambda) \oplus J_k\left(\frac{1}{\lambda}\right)\right)$.*

4.2. *A similar to $\overline{A}$*

Suppose $A \in M_n$ is similar to $\overline{A}$. The following theorem shows that the matrix of similarity may be chosen to be coninvolutory. Hence, if a nonsingular matrix A is similar to $\overline{A}$, we prove in the following theorem that A is a ψ_S-orthogonal matrix for some $S \in \mathscr{C}_n$.

Theorem 30. *Let $A \in M_n$ be given. The following are equivalent*:

(a) *A is similar to $\overline{A}$.*
(b) *A is similar to a real matrix $R \in M_n$.*
(c) *A is similar to a real R via a coninvolutory matrix.*
(d) *A is similar to $\overline{A}$ via a coninvolutory matrix.*

Proof. Since any matrix A is similar to its transpose, if A is similar to $\overline{A}$, then A is similar to A^*. By Theorem 4.1.7 of [1], A is similar to a real matrix, hence (a) implies (b).

Suppose $A = X^{-1}RX$ for some real R and a nonsingular X. By the real-coninvolutory decomposition, there exists a real S and a coninvolutory E such that $X = SE$. Then $A = E^{-1}S^{-1}RSE$ and $S^{-1}RS$ is real, hence (c) follows.

Suppose $A = E^{-1}RE$ for a real R and a coninvolutory E. Then $\overline{A} = \overline{E}^{-1}\overline{RE} = ERE^{-1}$, and thus $R = E^{-1}\overline{A}E$. This implies $A = (E^2)^{-1}\overline{A}E^2$, that is, A is similar to $\overline{A}$ via a coninvolutory matrix and thus (c) implies (d).

One checks that (d) implies (a). □

Suppose $A \in M_n$ is similar to $\overline{A}$. Then whenever $J_k(\lambda)$ is in the Jordan canonical form of A, so is $J_k(\overline{\lambda})$. Notice that $J \equiv J_k(\lambda) \oplus J_k(\overline{\lambda})$ is similar to $\overline{J}$ via the skew-coninvolutory matrix J_{2k}. Using arguments similar to Theorem 24, we obtain the following class of matrices which satisfies Theorem 30.

Theorem 31. *Let $A \in M_n$ have Jordan canonical form $\oplus(J_k(\lambda) \oplus J_k(\overline{\lambda}))$. Then A is similar to $\overline{A}$ via a skew-coninvolutory matrix.*

If A is similar to $\overline{A}$ and λ is a real eigenvalue of A, then the Jordan blocks corresponding to λ need not come in pairs. Thus begs the question whether A is similar to $\overline{A}$ via a skew-coninvolutory matrix if there is an unpaired Jordan block $J_k(\lambda)$ for some $\lambda \in \mathbb{R}$. Observe that if $A \in M_{n,k}$ and $\mu \in \mathbb{C}$ such that $AJ_k(\lambda) = \overline{J_n(\mu)}A$, then A would be of the form given in Eq. (7). Hence we have the following analogous results for the case when A is similar to $\overline{A}$.

Lemma 32. *Let $J \in M_n$ such that $J = \oplus_{r=1}^{m} J_r$, where $J_r \in M_{n_r}$,*

$$J_r = \begin{cases} \oplus_j J_{k_{rj}}(\lambda_r) & \text{if } \lambda_r \in \mathbb{R}, \\ \oplus_j [J_{k_{rj}}(\lambda_r) \oplus J_{k_{rj}}(\overline{\lambda_r})] & \text{if } \lambda_r \notin \mathbb{R}, \end{cases}$$

the λ_r's are distinct and $\lambda_r \neq \overline{\lambda_s}$ if $r \neq s$. If $X \in M_n$ such that $XJ = \overline{J}X$, then $X = \oplus_{r=1}^{m} X_r$, where $X_r \in M_{n_r}$.

Lemma 33. *Let $\lambda \in \mathbb{R}$ and let $J = \oplus_{i=1}^{r} J_{k_i}(\lambda) \in M_n$ such that there is an unpaired block $J_{k_i}(\lambda)$ for some i. Then J is not similar to $\overline{J}$ via a skew-coninvolutory matrix.*

Theorem 34. *$A \in M_n$ is similar to $\overline{A}$ via a skew-coninvolutory matrix if and only the Jordan canonical form of A is $J = \oplus(J_k(\lambda) \oplus J_k(\overline{\lambda}))$.*

Thus A is a ψ_S-orthogonal matrix for some $S \in \mathscr{D}_n$ if and only if its Jordan canonical form consists of pairs of $J_k(\lambda) \oplus J_k(\overline{\lambda})$.

4.3. A similar to $-\overline{A}$

We consider a related result for the case when $A \in M_n$ is similar to $-\overline{A}$.

Theorem 35. *Let* $A \in M_n$. *The following are equivalent*:

(a) A *is similar to* $-\overline{A}$.
(b) $A = X^{-1}PX$, *where* P *is pure imaginary.*
(c) $A = E^{-1}PE$, *where* E *is coninvolutory and* P *is pure imaginary.*
(d) $A = E^{-1}(-\overline{A})E$, *where* E *is coninvolutory.*

Proof. If A is similar to $-\overline{A}$, then whenever $J_k(\lambda)$ occurs in the Jordan canonical form of A, so does $J_k(-\overline{\lambda})$. If $\lambda = ia$ for some real a, then $\lambda = -\overline{\lambda}$. Hence, the Jordan blocks in the Jordan canonical form of A are $J_k(ia)$ for some $a \in \mathbb{R}$ or $J_k(\lambda) \oplus J_k(-\overline{\lambda})$, whenever λ is not pure imaginary and $J_k(ia)$ is similar to the pure imaginary matrix $iJ_k(a)$. On the other hand, since

$$\begin{bmatrix} iI_k & -I_k \\ -I_k & iI_k \end{bmatrix}\begin{bmatrix} J_k(\lambda) & 0 \\ 0 & -J_k(\overline{\lambda}) \end{bmatrix}\begin{bmatrix} -iI_k & -I_k \\ -I_k & -iI_k \end{bmatrix} = \begin{bmatrix} J_k(\lambda) - J_k(\overline{\lambda}) & -i(J_k(\lambda) + J_k(\overline{\lambda})) \\ i(J_k(\lambda) + J_k(\overline{\lambda})) & J_k(\lambda) - J_k(\overline{\lambda}) \end{bmatrix},$$

which is pure imaginary, then $\oplus(J_k(\lambda) \oplus J_k(-\overline{\lambda}))$ is similar to a pure imaginary matrix. Hence $A = X^{-1}PX$, for some nonsingular X and a pure imaginary P.

Suppose $A = X^{-1}PX$, where P is pure imaginary. Let $X = RE$ be a real-coninvolutory decomposition of X. Then $A = E^{-1}(R^{-1}PR)E$, and $R^{-1}PR$ is still pure imaginary.

Suppose $A = E^{-1}PE$, where E is coninvolutory and P is pure imaginary. Then $\overline{A} = \overline{E}^{-1}\overline{PE} = -EPE^{-1}$ which implies that $P = E^{-1}(-\overline{A})E$. Thus, $A = E^{-1}PE = (E^2)^{-1}(-\overline{A})E^2$. Therefore, A is similar to $-\overline{A}$ via a coninvolutory matrix.

One checks that (d) implies (a). □

Thus, if A is a nonsingular matrix similar to $-\overline{A}$, then, by Theorem 35, $\psi_S(A) = -A^{-1}$ for some $S \in \mathscr{C}_n$. We present a class of matrices satisfying Theorem 35.

Theorem 36. *Let* $A \in M_n$ *have Jordan canonical form* $\oplus(J_k(\lambda) \oplus J_k(-\overline{\lambda}))$. *Then* A *is similar to* $-\overline{A}$ *via a skew-coninvolutory matrix.*

Proof. Observe that $J_k(\lambda) \oplus -J_k(\overline{\lambda}) = J_{2k}\left[-\overline{(J_k(\lambda) \oplus -J_k(\overline{\lambda}))}\right]J_{2k}^{-1}$. Using similar arguments in the proof of Theorem 24, we conclude that A is similar to $-\overline{A}$ via a skew-coninvolutory matrix. □

If A is similar to $-\overline{A}$ and $\lambda \in \mathbb{R}$i is an eigenvalue of A, then the Jordan blocks corresponding to λ need not come in pairs. Suppose $X \in M_{n,k}$ such that $XJ_k(\lambda) = -\overline{J_n(\mu)}X$. Then X will be of the form given in Lemma 25. We follow the arguments in the case when A is similar to $\overline{A}^{-1}$ to prove the converse of Theorem 36.

Lemma 37. *Let* $J \in M_n$ *such that* $J = \oplus_{r=1}^{m} J_r$, *where* $J_r \in M_{n_r}$,

$$J_r = \begin{cases} \oplus_j J_{k_{rj}}(\lambda_r) & \text{if } \lambda_r \in \mathrm{i}\mathbb{R}, \\ \oplus_j [J_{k_{rj}}(\lambda_r) \oplus J_{k_{rj}}(-\overline{\lambda_r})] & \text{if } \lambda_r \notin \mathrm{i}\mathbb{R}, \end{cases}$$

the λ_r*'s are distinct and* $\lambda_r \neq -\overline{\lambda_s}$ *if* $r \neq s$*. If* $X \in M_n$ *such that* $XJ = -\overline{J}X$*, then* $X = \oplus_{r=1}^{m} X_r$*, where* $X_r \in M_{n_r}$*.*

Lemma 38. *Let* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *and let* $J = \oplus_{i=1}^{r} J_{k_i}(\lambda) \in M_n$ *such that there is an unpaired block* $J_{k_i}(\lambda)$ *for some* i*. Then* J *is not similar to* $-\overline{J}$ *via a skew-coninvolutory matrix.*

Theorem 39. $A \in M_n$ *is similar to* $-\overline{A}$ *via a skew-coninvolutory matrix if and only the Jordan canonical form of* A *is* $J = \oplus(J_k(\lambda) \oplus J_k(-\overline{\lambda}))$*.*

Theorem 39 implies that a nonsingular matrix A satisfies $\psi_S(A) = -A^{-1}$ for some $S \in \mathscr{D}_n$ if and only if the Jordan canonical form of A consists of pairs of $J_k(\lambda) \oplus J_k(-\overline{\lambda})$.

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